How to take shortcuts in Euclidean space: making a given set into a short quasi-convex set

Jonas Azzam and Raanan Schul

December 7, 2009

Contents

Abstract

For a given connected set Γ in $d-$dimensional Euclidean space, we construct a connected set $\tilde{\Gamma} \supset \Gamma$ such that the two sets have comparable Hausdorff length, and the set $\tilde{\Gamma}$ has the property that it is quasiconvex, i.e. any two points x and y in $\tilde{\Gamma}$ can be connected via a path, all of which is in $\tilde{\Gamma}$, which has length bounded by a fixed constant multiple of the Euclidean distance between x and y. Thus, for any set K in $d-$dimensional Euclidean space we have a set $\tilde{\Gamma}$ as above such that $\tilde{\Gamma}$ has comparable Hausdorff length to a shortest connected set containing K. Constants appearing here depend only on the ambient dimension d. In the case where Γ is Reifenberg flat, our constants are also independent the dimension d, and in this case, our theorem holds for Γ in an infinite dimensional Hilbert space. This work closely related to $k-$spanners, which appear in computer science. **Keywords:** chord-arc, quasiconvex, k-spanner, traveling salesman.

1 Statement of main theorem

For a curve γ in $\mathbb{R}^d$, let $\ell(\gamma)$ denote the arclength of γ. For a set $K \subset \mathbb{R}^d$, let $\mathscr{H}^1(K)$ denote the $1-$dimensional Hausdorff measure of K. We prove the following theorem.

Theorem 1. *Let $d \geq 2$. There exist constants $C_1, C_2 > 1$, C_1 depending on d, such that for any subset $K \subset \mathbb{R}^d$ there exists a connected set $\tilde{\Gamma} \subset \mathbb{R}^d$ such that:*

(i) $\tilde{\Gamma} \supset K$.

(ii) $\mathscr{H}^1(\tilde{\Gamma}) \leq C_1 \mathscr{H}^1(\Gamma)$ *for any connected* $\Gamma \supset K$.

(iii) For any $x, y \in \tilde{\Gamma}$ there is a path connecting x and y, $\gamma_{x,y} \subset \tilde{\Gamma}$, with

$$\ell(\gamma_{x,y}) \leq C_2 |x - y| \,.$$

A set $\tilde{\Gamma}$ satisfying property (iii) above above is called quasiconvex.

The case $d = 2$ was first shown by Peter Jones while proving his Traveling Salesman Theorem (see [9]) using complex analysis.

Let us mention a relation to computer-science. For a (possibly weighted) graph $G = (V, E)$, a k-spanner is a subgraph with the same vertices, $G' = (V, E')$, in which every two vertices are at most k times as far apart on G' (in the graph metric) than on G. This is a useful concept in studying network optimization. A geometric

$k-$spanner is a graph over a set of vertices K in Euclidean space, such that the graph distance is bounded by k times the Euclidean distance for any two points in K. See [12, 13] for more details on how these are useful in computer science. We note that the problem we are dealing with is harder than finding k-spanners. For a given set K, we are concerned with finding a 'not too long' set $\tilde{\Gamma}$, such that $\tilde{\Gamma}$ is a geometric $k-$spanner for itself, not just for the set K, in particular we are building a network which is not too long, and in which all new nodes are also well connected. (Also note that in our case, we must also treat the edges as continua of nodes.)

The Traveling Salesman Theorem is a major tool used in our proof (see Theorem 6 below). It holds in the setting of an infinite dimensional Hilbert space. This is one reason why the authors believe the following.

Conjecture 2. *Theorem 1 holds with constants independent of dimension and in fact holds in the for the case where K is a subset of an infinite dimensional Hilbert space.*

See Remark 11 for a discussion of where our present proof breaks down in this context. Under some flatness assumptions we can say more. A set K is called $\varepsilon-$Reifenberg flat (with holes) if for any ball B of radius r, we have that $K \cap B$ is contained inside a tube of radius εr, where $\varepsilon > 0$ is some fixed constant. Our proof of Theorem 1, coupled with the proof of Theorem 6, yields the following theorem. (We, unfortunately, must appeal to the proof of Theorem 6, and not its statement.)

Theorem 3. *There exist constants $C_1, C_2 > 1$, and $\varepsilon > 0$ such that for any ε-Reifenberg flat (with holes) set $K \subset \mathscr{H}$, a (possibly infinite dimensional) Hilbert space, there exists a connected set $\tilde{\Gamma} \subset \mathbb{R}^d$ such that:*

(i) $\tilde{\Gamma} \supset K$.

(ii) $\mathscr{H}^1(\tilde{\Gamma}) \leq C_1 \mathscr{H}^1(\Gamma)$ *for any connected* $\Gamma \supset K$.

(iii) For any $x, y \in \tilde{\Gamma}$ there is a path connecting x and y, $\gamma_{x,y} \subset \tilde{\Gamma}$, with

$$\ell(\gamma_{x,y}) \leq C_2 |x - y| \,.$$

We note that the work presented in this paper is not the first extension of the $d = 2$ version of Theorem 1. The following theorem holds for any Hilbert space.

Theorem 4 ([9, 7, 1])**.** *There is $M > 0$ such that if Γ is a rectifiable simple closed curve in a Hilbert space $\mathscr{H}$, then there is a collection $\{C_j\}$ of M chord-arc curves of positive length in $\mathscr{H}$ such that $\bigcup C_j$ is connected,*

$$\sum \ell(C_j) \leq M \mathscr{H}^1(\Gamma) \tag{1}$$

and

$$\mathscr{H}^1(\Gamma \setminus \bigcup C_j) = 0. \tag{2}$$

This result was proved for the plane by Peter Jones (see [9]); for a finite dimensional Hilbert space by John Garnett, Peter Jones, and Donald Marshall by adapting the analytic techniques of Jones' original argument to the minimal surface spanned by Γ (see [7]); for an infinite dimensional Hilbert spaces this was shown by the first author, using a limiting argument (see [1]).

Other related works are, for example, [10] and [5]. Kenyon and Kenyon [10] is a mathematically weaker version of Theorem 1 for $d = 2$, which has the advantage that is computationally tractable. Das and Narasimhan, in [5], improve on [10] and extend to $d > 2$. Both of these fit within the k-spanner setting in that they are concerned only with the well-connectedness of nodes in the original set K, and not with the well-connectedness of the resulting set. Christopher Bishop, in [2], actually improves on Theorem 1 for $d = 2$. This work also has the advantage of being computationally tractable.

1.1 Organization

The paper is organized as follows. In Section 2 we set-up some notation and tools we will use. In particular we denote by Γ a connected set of shortest Hausdorff length containing K. In Section 3 we add the needed paths to Γ, giving us a connected set $\tilde{\Gamma}$ which does not have length more than a costant times that of Γ. In Section 4 we show that $\tilde{\Gamma}$ satisfies the properties of Theorem 1, in particular, that any points x and y in $\tilde{\Gamma}$ can be connected via a path, all of which is in $\tilde{\Gamma}$, which has length bounded by a fixed constant multiple of the Euclidean distance between x and y.

1.2 Acknowledgements

The authors would like to thank the Centre de Recerca Matemàtica, Barcelona for holding a conference where they developed some of the main ideas for this paper, and John Garnett for his helpful advice. The second author was supported in part by NSF DMS 0502747 and NSF DMS 0800837 (renamed to NSF DMS 0965766).

1.3 Animation

The first author created some animation exemplifying the construction in this paper. It is available at
`http://www.math.sunysb.edu/~schul/math/AzzamSC-link.html`

2 Notation and tools

2.1 Notation

Let $|A|$ denote the diameter of a set A. Let $\mathscr{H}^1(A)$ denote the 1-dimensional Hausdorff measure of A and for a curve γ, let $\ell(\gamma)$ denote the arclength of γ. See [11] for a discussion of Hausdorff measure and arclength. For a set $A \subseteq \mathbb{R}^d$, define

$$A_\delta = \{x \in \mathbb{R}^d : \text{dist}(x, A) < \delta\}.$$

For points $x, y \in \mathscr{H}$, and $\rho \geq 0$, we will define

$$R_\rho(x, y) := B(\frac{x+y}{2}, \frac{1+\rho}{2}|x+y|)$$

and let $R(x, y) := R_0(x, y)$. Also define

$$S_\lambda(x, y) = B(x, (1-\lambda)|x-y|) \cap B(y, (1-\lambda)|x-y|)$$

and let $S(x, y) := S_0(x, y)$.

For a ball B, and a set K, define the Jones$-\beta$ number, $\beta_K(B)$ by setting $\beta_K(B)|B|$ to be the width of the smallest tube containing $K \cap B$, i.e.

$$\beta_K(B) := \inf_{L \text{ line}} \sup_{x \in B \cap K} \frac{\text{dist}(x, L)}{|B|}.$$

We often omit the subscript K when it is clear from context. For $M > 0$, we let MB denote the ball with the same center but diameter $M|B|$.

If γ is a curve with initial and terminal points x and y, we say that γ is a *chord-arc path with constant C*, if its arclength parametrization is a C-bilipschitz function. If we do not specify C, we assume it is obvious from the context. In this paper, we will be constructing chord-arc-paths with constant C_2, where C_2 is a sufficiently large constant to be determined later.

Let Γ be a connected set containing K. We may assume $\mathscr{H}^1(\Gamma) < \infty$.

2.2 Cones

For any point ξ in a set A, define $C_{\alpha,A}$ the (α, A) *cone with apex* ξ to be the union of connected components of the set

$$\{z \in \mathbb{R}^d : |z - \xi| < \alpha \text{dist}(z, \Gamma)\} \tag{3}$$

which contain ξ in their closures (the case of their being more than one such component is most evident in two dimensions when, say, Γ is a circle, although this may still occur in higher dimensions; see Figure 3). We will let $C_\alpha(\xi) = C_{\alpha,\Gamma}(\xi)$.

2.3 Nets, Grids, and Cubes

Let $\{\Delta_k\}_{k>0}$ be an increasing sequence of 2^{-k}-nets in Γ, and assume $|\Gamma|$ is small enough so that $\Delta_0 = \{\xi_0\}$. Such a sequence may be constructed via induction on k.

Here we create a lattice in the complement of Γ that mimics a Whitney decomposition. Let k_0 be an integer to be chosen later. Let $\mathscr{N}_0$ be a 2^{-k_0}-net for $\Gamma_1 \backslash \Gamma_{1/2}$, define $\mathscr{N}_k'$ to be a 2^{-k-k_0}-net for $\left((\Gamma_{2^{-k}} \backslash \Gamma_{2^{-k-1}}) \bigcup \mathscr{N}_{k-1} \right)$ containing $\mathscr{N}_{k-1}$. Let $\mathscr{N}_k = (\Gamma_{2^{-k}} \backslash \Gamma_{2^{-k-1}}) \cap \mathscr{N}_k'$. Let $\mathscr{N} = \bigcup \mathscr{N}_k$. The set $\mathscr{N}$ forms the vertices of a "grid" in the complement of Γ upon which we will build our bridges by constructing polygonal paths between nearby points in $\mathscr{N}$. This will ensure that the angles between segments in each path don't become too small. We note that, for k_0 large enough, we may ensure that

$$x \in \mathscr{N}_k \text{ and } y \in \mathscr{N} \cap B(x, 10 \cdot 2^{-k-k_0}) \Rightarrow y \in \mathscr{N}_{k-1} \cup \mathscr{N}_k \cup \mathscr{N}_{k+1}. \tag{4}$$

Also note that for all $k > 0$, every point in $\Gamma_{2^{-k}}$ is within 2^{-k-k_0+2} of a point in $\mathscr{N}$. Let $\mathscr{B}_0 = \{B_0\} = \{B(\xi_0, 1)\}$ and for $k > 0$,

$$\mathscr{B}_k := \{B(\xi, 2^{-k}) : \xi \in \Delta_k\}, \text{ and } \mathscr{B} = \bigcup \mathscr{B}_k.$$

Note that $B \in \mathscr{B}_k$ implies $\frac{1}{2}B \in \mathscr{B}_{k+1}$.

We need a version of dyadic cubes in the spirit of Michael Christ or Guy David. We do not have an underlying measure, so we cannot appeal to their constructions, however we can use ideas from [4, 6]. We fix a constant $J = 100$, and give a family (i.e. tree) structure on $\cup_k \Delta_{kJ}$. For each $x \in \Delta_{kJ}$ where $k > 0$, we define a unique parent $y \in \Delta_{(k-1)J}$, so that $|y - x|$ is minimized. If there is more than one such possible y, choose randomly. By the construction of Δ_{kJ}, we have that $2^{-kJ} \le |y - x| < 2^{-kJ+1}$. Let $D(x)$ be the collections of descendants of x by the above family relation, and set $D_j(x) = D(x) \cap \Delta_j$, where j here satisfies $j = Jl$ for some $l \ge 0$. [RS: above, in a few places, i removed refrences to a family structure on all scales. now it is just for scales which are multiples of J.] For $k \ge 0$, and $x \in \Delta_{kJ}$, let

$$Q^o(x) = \bigcup_{l \ge 0} \bigcup_{\substack{z \in D_j(x) \\ j = (l+k)J}} B(z, 2^{-j-100}),$$

and let $Q(x)$ be the closure of $Q^o(x)$. Let $\mathscr{Q}_k = \{Q(x) : x \in \Delta_k\}$. We have have the properties described below.

Lemma 5. *For $k \ge 0$ we have the following.*

(i) $\Gamma = \cup_{x\in\Delta_{kJ}} Q(x)$.

(ii) *If* $x_1, x_2 \in \Delta_{kJ}$, *then* $Q^o(x_1) \cap Q^o(x_2) = \emptyset$.

(iii) *If* $x \in \Delta_{kJ}$ *then* $B(x, 2^{-kJ-100}) \subset Q^o(x) \subset Q(x) \subset B(x, 2^{-kJ}(1 + 2^{-J}))$.

(iv) *If* $l \geq k$, $y \in \Delta_l$, *and* $Q^o(x) \cap Q^o(y) \neq \emptyset$ *then* $Q^o(x) \supset Q^o(y)$.

We note that if $\Gamma \subset \mathbb{R}^d$ then the quantity $D_j(x)$ grows exponentially in j, with fixed base, depending on the dimension d. We will not make use of this fact, nor will we need any bound on this growth except under special circumstances, where we will have explicit bounds which will be independent of d.

Proof. First, (i) follows from the definition and induction on k. To see (ii), suppose that $y \in Q^o(x_1)$ and assume that $y \in D(x_1)$. Then, $\mathrm{dist}(x, y) \leq \sum_{p=0}^{l-1} 2^{-pJ}2^{-kJ} \leq 2^{-kJ}(1 + 2^{-J})$. Furthermore, if $x' \in D(x_1) \cap D_{(k+1)J}$ with $y \in D(x')$, then

$$\mathrm{dist}(x', x_2) \geq \mathrm{dist}(x', x_1)\,,$$

and by the triangle inequality, $2\mathrm{dist}(x', x_2) \geq \mathrm{dist}(x_1, x_2) \geq 2^{-kJ}$ giving us $\mathrm{dist}(x', x_2) \geq 2^{-kJ-1}$ Also, as above, $\mathrm{dist}(x', y) \leq 2^{-kJ-J}(1 + 2^{-J})$. Using the triangle inequality again,

$$\mathrm{dist}(y, x_2) \geq 2^{-kJ-1} - 2^{-kJ-J}(1 + 2^{-J}) > 2^{-kJ-1}(1 - 2^{-J})$$

for $J > 2$, for example $J = 100$. This gives that $y \notin B(x_2, 2^{-kJ-100+1})$. We can run the same argument for any other ball in the definition of $Q^o(x_2)$. In particular, we get (ii) by the density of $D(x_1)$ in $Q^o(x_1)$. Similarly, (iii) and (iv) follow as well. $\square$

We denote by $MQ(x)$ the set $\{x : \mathrm{dist}(x, Q(x)) \leq M\mathrm{diam}(Q(x))\}$ and for a ball $B \in \mathscr{B}$, $Q(B) \in \mathscr{Q}$ is the smallest cube containing it. We will assume $M > 2^{J+2}$.

2.4 The Traveling Salesman Theorem

The last tool we use is the following theorem:

Theorem 6 ((Analyst's) traveling Salesman Theorem)**.** *For a set* K *in a Hilbert space* $\mathscr{H}$, *define*

$$\beta_K = |K| + \sum_k \sum_{B\in\mathscr{B}_k} \beta_K^2(MB)|B|.$$

There is M_0 such that for $M > M_0$ and any set K, if β_K is finite, then K may be contained in a connected set Γ such that

$$\mathscr{H}^1(\Gamma) \lesssim \beta_K .$$

Moreover, if Γ is any rectifiable set of finite length, then

$$\beta_\Gamma \lesssim \mathscr{H}^1(\Gamma)$$

for any $M > 1$.

Note that this imples that if $\Gamma \subset \mathscr{H}$ is a connected set, then

$$\beta_\Gamma \sim \mathscr{H}^1(\Gamma)$$

This theorem was originally proved for $\mathscr{H} = \mathbb{C}$ by Peter Jones [9], then generalized to $\mathscr{H} = \mathbb{R}^d$ by Kate Okikiolu [14], and to infinite dimensional Hilbert spaces by the second author of this paper [15].

We now begin the proof of the main theorem by showing we may contain Γ in a set $\tilde{\Gamma}$ satisfying

$$\mathscr{H}^1(\tilde{\Gamma}) \lesssim \mathscr{H}^1(\Gamma) + \beta(\Gamma) \lesssim \mathscr{H}^1(\Gamma) .$$

Remark 7. By the proof of the traveling Salesman Theorem, we may assume, by allowing an increase of $\mathscr{H}^1(\Gamma)$ by a constant multiple, that Γ satisfies the following properties for balls B with center in Γ and $\beta(2B) < \varepsilon$, with ε sufficiently small

- There is a component of $\Gamma \cap B$ with diameter at least $|B|(1-\varepsilon)$.
- The Hausdorff distance between $\Gamma \cap B$ and $L \cap B$ is bounded by $4\varepsilon|B|$ for some affine line L.

Furthermore, if Γ had initially been $\varepsilon-$Reifenberg flat with holes, then the above may be achieved while keeping Γ $2\varepsilon-$Reifenberg flat; if K is $\varepsilon-$Reifenberg flat with holes, then one may construct $\Gamma \supset K$ $2\varepsilon-$Reifenberg flat and such that $\mathscr{H}^1(\Gamma) \lesssim \mathscr{H}^1(\Gamma')$ for any $\Gamma' \supset K$. Henceforth, we shall assume Γ has these properties.

2.5 Outline

Let us give a rough idea of our plan. The proof of the theorem is a stopping time process run on a family of balls centered along Γ. The idea is that when a certain stopping time function becomes too big on one of the balls, this tells us to build a

bridge between points. At first, it would seem that we merely have to check when the β-number of a ball was too large since this would detect a bend in the curve Γ where one should build a short-cut. However, this doesn't account for the case of sets which have small β on all scales but contain a lot of length. (It is amusing to note that if we didn't have the assumption that Γ had finite length, then Γ could possibly satisfy $\beta(MB) \sim \varepsilon$ for all sufficiently small balls B with centers in Γ, in which case all balls will be "flat" or have small β, but Γ will have dimension at least $1 + c\varepsilon^2$ for some constant c independent of ε; see exercises in chapter X of [8]). Therefore, it is necessary to keep a history of the β-numbers through the stopping time process, that is, not only do we keep track of the β of a ball but also of the balls in the previous generations containing it (see condition (6) below). We run the stopping time process until a chain of balls have accumulated a large total amount of β-numbers, and in this event we add a bridge. Separate treatment is given to balls with $\beta(MB)$ bounded away from zero.

3 Constructing shortcuts

In this section we will classify all balls into three classes and explain how we build bridges in each of those cases. We will record some of their properties, and use those in Section 4.

3.1 The Bridges

The general idea for building a bridge between two points in Γ inside a ball B is to pick a point $z \in \Gamma^c$ whose distance from each of those points is $\sim |B|$ and then connect it to both of those points. This is not as trivial as it sounds. If one is not careful enough, it may be the case that after adding all our bridges to Γ to form $\tilde{\Gamma}$, while each pair of points in Γ may be joined by a path of small relative length, points between the bridges themselves may have to travel a long relative distance to reach each other; imagine two bridges connecting two different pairs of points in Γ, but their middles being very close (i.e. they form a narrow overpass). Building our bridges as polygonal paths with vertices in $\mathscr{N}$ will help guarantee that points on different bridges can only be as close as their distance from Γ.

Lemma 8. *Suppose $\xi \in \Gamma$, $z \in \mathscr{N}_k \cap B(\xi, C2^{-k}) \cap C_\alpha(\xi)$, ξ is in the closure of some component A of $C_\alpha(\xi)$, and z and ξ may be joined by a path p in A. Moreover, suppose p has the property that for any ball $B = B(w, 2^{-j-k_0+1})$ with $w \in \mathscr{N}_j$ that intersects p, $2B \cap p$ is connected. Then there is a path p' connecting z and ξ with the following properties:*

a. p' is a polygonal path in $C_{2\alpha}(\xi)$ with vertices in $\mathscr{N}$

b. $\ell(p') \lesssim \ell(p)$,

c. if $[x,y]$ *is an edge in the path* p' *and* $x \in \mathscr{N}_j$ *for some* j*, then* $y \in \mathscr{N}_{j-1} \cup \mathscr{N}_j \cup \mathscr{N}_{j+1}$ *and there is* $\lambda > 0$ *such that*

$$R_\lambda(x,y) \cap (\mathscr{N} \backslash \{x,y\}) = \emptyset\,,$$

and

d. $|x-y| \sim 2^{-j-k_0}$*. Here,* λ *and all other implied constants are universal.*

Proof. Since $\{B(w, 2^{-j-k_0+1}) : w \in \mathscr{N}_j, j \geq 0\}$ is a cover of Γ^c, consider the subcollection $\mathscr{C}$ of all those balls that intersect $p\backslash\{\xi\}$. Let $\gamma : [0, \ell(p)) \to p\backslash\{\xi\}$ be the arclength parametrization of $p\backslash\{\xi\}$, Let $\mathscr{I} = \{\gamma^{-1}(2B) : B \in \mathscr{C}\}$. Choose a subcollection $\{I_j\} \subseteq \mathscr{I}$ so that the right most endpoint of I_j is contained in I_{j+1} and so that $I_0 = \gamma^{-1}(B(z, 2^{-k-k_0+2}))$, and so that no point in $[0, \ell(p))$ is contained in more than two sets in $\{I_j\}$. Hence, $\sum \ell(I_j) \lesssim \ell(p)$. Let $B_j \in \mathscr{C}$ be the ball so that $I_j = \gamma^{-1}(2B_j)$ and let w_j be their centers, with $w_0 = 0$. Then $2B_j \cap 2B_{j+1} \neq \emptyset$ and by (4), $|w_j - w_{j+1}| \lesssim |B_j|$.

$$\sum |B_j| \lesssim \sum \ell(I_j) \lesssim \ell(p).$$

Let p' be the path $\bigcup_{j=0}^{\infty} [w_j, w_{j+1}]$. Then

$$\ell(p') \leq \sum |w_j - w_{j+1}| \leq \sum 2|B_j| \lesssim \ell(p).$$

Hence we can find a polygonal path so that (b) is satisfied. We will now adjust this path so that (b) is still satisfied but so that (c) is true. Let $[x,y]$ be an edge in p'. If $x \in \mathscr{N}_j$, by the work above, $y \in B(x, 2^{-j-k_0+3})$ and hence is in $\mathscr{N}_{j-1} \cup \mathscr{N}_j \cup \mathscr{N}_{j+1}$ and $|x-y| > 2^{-j-k_0-1}$. Let $r = |x-y|$, then by the previous sentence there is a constant $b = b(k_0) < 1$, such that $(B(x,br) \cup B(y,br)) \cap \mathscr{N} = \emptyset$. By some planar geometry, (see Figure 1), there exists a small constant $\lambda = \lambda(b) > 0$ such that

$$R_\lambda(x,y) \subseteq B(x,cr) \cup B(y,cr) \cup S_\lambda(x,y).$$

Suppose that $[x,y]$ didn't satisfy (c). Let $w \in R_\lambda(x,y) \cap (\mathscr{N}\backslash\{x,y\})$. Replace the edge $[x,y]$ with $[x,w]$ and $[w,y]$. The total length we have added is no more than $[x,y]$, and moreover, since $w \notin B(x,br) \cup B(y,br)$, we have $|w-x|, |w-y| < (1-\lambda)|x-y|$. Repeat the process on these two new edges, checking to see if they satisfy (c) and replacing them if not. This replacement can only happen a finite number of times, since the vertices of any new edge we add must be in $\mathscr{N}_{j-1} \cup \mathscr{N}_j \cup \mathscr{N}_{j+1}$ by (4), but their mutual distances are decreasing by a factor

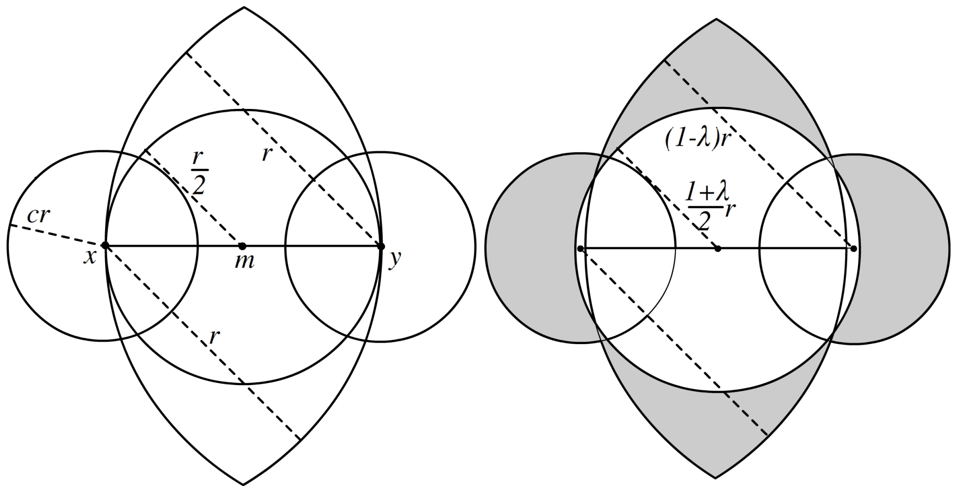

Figure 1: On the left are the balls $B(x, cr), B(y, cr), R(x, y)$, and the set $S(x, y)$. Then by slightly adjusting r in the definition of the latter two sets, we know there is λ small such that $R_\lambda(x, y)$ is contained in the shaded region $B(x, cr) \cup B(y, cr) \cup S_\lambda(x, y)$.

of $1 - \lambda$ each time we add a new edge. By doing this on each edge in p' a finite number of times, we have adjusted p' into a path that satisfies (c) and increased it's length by no more than some universal factor.

Suppose $|x - y|$ is an edge with $x \in \mathscr{N}_j$. We have already seen that $|x - y| \gtrsim 2^{-k-k_0}$. If $|x - y| \geq 10 \cdot 2^{-k-k_0}$, then $B(\frac{x+y}{2}, 2^{-k-k_0+2}) \cap \mathscr{N}$ is empty by part (c), but this contradicts the sentence following (4).

Finally, if k_0 is large enough (depending on α), the final product p' will be contained in $C_{2\alpha}(\xi)$, which gives (a). $\square$

Remark 9. We will only replace paths p with polygonal paths p' when p satisfies the conditions of this lemma. In fact, it so happens that the only paths p we ever have to deal with are polygonal paths composed of either one segment or two segments that make an angle of $\frac{\pi}{4}$. Hence, when we refer to a path or polygonal path, we will assume it has the properties in the lemma.

3.2 Three types of cubes

We will classify all cubes in $\mathscr{Q}$ as Flat-Good, Flat-Bad, or Non-Flat. Let $\delta, \varepsilon > 0$ and $M > 0$, to be chosen later (see subsection 4.2). If $B \in \mathscr{B}$ satisfies $\beta(MB) > \delta\varepsilon$ we say that B is of *Non-Flat*. If $Q = Q(x) \in \mathscr{Q}_k$ is such that $B(x, 2^{-kJ})$

is Non-Flat, call Q *Non-Flat* as well. The rest of the cubes are divided into two distinct classes:Flat-Good and Flat-Bad. The class Flat-Bad will be defined in the following section, and the class Flat-Good will simply be all the cubes in $\mathscr{Q}$ which are not of the types Flat-Bad or Non-Flat.

3.2.1 Definition of and construction at Flat-Bad cubes

Let $\varepsilon > 0$ and $M > 0$, to be chosen later, with $\varepsilon \ll \frac{1}{M}$ (see subsection 4.2). Suppose Γ has diameter small enough so that $\beta(Q_0) < \varepsilon$, where $Q_0 = Q(\xi_0)$, $\Delta_0 = \{\xi_0\}$. Call $\{Q_j\}_{j=m}^k$ a *chain* if $Q_j \in \mathscr{Q}_j$, Q_{j+1} is a child of Q_j for each $j \geq m$.

Going through the cubes in order, consider the first cube Q (if it exists) such that there is a chain $\{Q_j\}_{j=0}^k$ so that $Q_k = Q \in \mathscr{Q}_k$, $\beta(Q_j) < \varepsilon\delta$ for $0 < j \leq k$ and

$$\sum_{j=1}^{k-1} \beta(MQ_j)^2 \leq \varepsilon < \sum_{j=1}^{k} \beta(MQ_j)^2. \tag{5}$$

This sum is essentially a truncated Jones function (see [8]). Call Q_k *Flat-Bad*. Let $B = B_k = B(x, 2^{-kJ})$ where x is the center of Q. Pick $z \in 2B_k \cap \mathscr{N}_{kJ}$ closest to the center of B.

For each $\xi \in 2B \cap \Delta_{kJ}$ (there will be at most three of these for ε small), pick $\xi' \in B(\xi, M\varepsilon|B|) \cap \Gamma$ that is closest to z, and note that, for small $\varepsilon \ll \frac{1}{M}$, $[z, \xi'] \subseteq C_\alpha(\xi')$, so by Lemma 8, we may connect z to ξ' by a polygonal path. See Figure 2.

Remark 10. Building our path within a component of $C_{2\alpha}(\xi')$ in this way ensures that the path won't intersect Γ at too sharp an angle or get too close to Γ before reaching ξ', and for this reason it is in fact necessary for any curves we add to be contained in cones centered on Γ.

It is also important to note why we can't necessarily connect z directly to ξ instead of the nearby point ξ'. In $\mathbb{R}^2$ this is evident since the point ξ' may be separated from z by Γ itself, making it impossible to connect z to ξ. In higher dimensions, it may be the case that we can connect z to ξ by a polygonal path, but possibly not without getting too close to Γ, which may be the case if Γ resembles a Peano curve near ξ. In other words, if we had simply taken $C_\alpha(\xi)$ to be the entire set in (3), it wouldn't always be possible to build our path in $C_\alpha(\xi)$ since whatever component we start building it in may not contain ξ in it's closure (see Figure 3). Hence, we will have to make due with building bridges to points ξ' near $\xi \in \Delta_k$.

We continue going through the tree in this manner. We stop and declare Q_k to

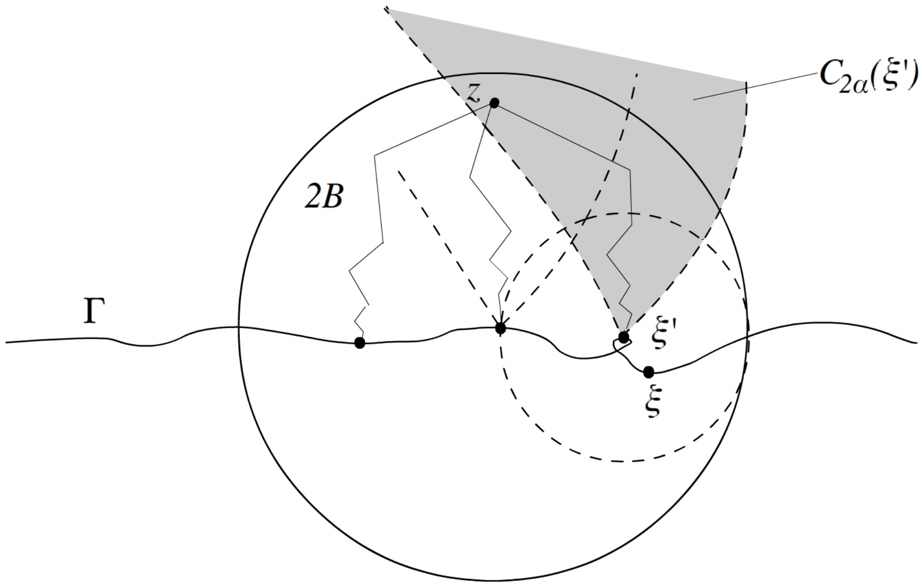

Figure 2: Building bridges for Flat-Bad balls.

be Flat-Bad every time we have Q_k satisfying

$$\sum_{j=m+1}^{k-1} \beta(MQ_j)^2 \leq \varepsilon < \sum_{j=m+1}^{k} \beta(MQ_j)^2 \tag{6}$$

for a chain of cubes $\{Q_j\}_{j=m}^{k}$, so that $\{Q_j\}_{j=m+1}^{k-1}$ are all not of type Non-Flat, where Q_m is one of: Q_0, a Non-Flat cube, or the previous Flat-Bad;

Let z be any point in $2B_k \cap \mathscr{N}_{kJ} \cap C_\alpha(\xi)$ closest to the center of B_k. For each $\xi \in \Delta_{kJ} \cap 2B_k$, we pick points $\xi' \in B(\xi, M\varepsilon 2^{-k})$ closest to z and connect them to z as before.

We remind the reader that if a cube is neither Flat-Bad or Non-Flat, call it *Flat-Good*. Note that all the intermediate cubes $\{Q_j\}_{j=m+1}^{k-1}$ in equation (6) are Flat-Good.

3.2.2 Definition of and construction at Non-Flat balls

Let $\delta > 0$, k_1, and $C > 2$ be numbers to be specified later. We recall that a ball $B = B(\xi_1, 2^{-k})$ is *Non-Flat* if $\beta(MB) > \delta\varepsilon$. In this case, consider all $\xi_2 \in B(\xi_1, C2^{-k}) \backslash B(\xi_1, 2^{-k+1}) \cap \Delta_k$ such that

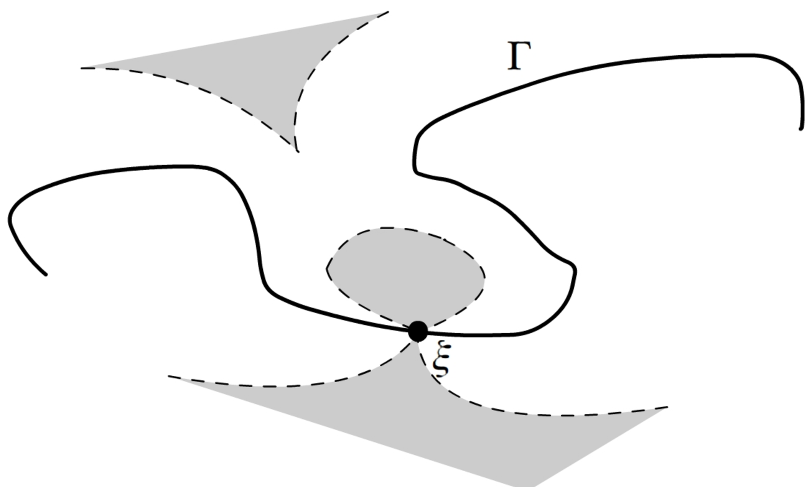

Figure 3: The shaded regions depict the set $\{z : |z - \xi| < \alpha d(z, \Gamma)\}$, which may have several components, some of which may not contain ξ in their boundaries.

(C1) $\exists \eta_j \in B(\xi_j, 2^{-k}) \cap \Gamma$, $j = 1, 2$, and $z \in C_{C\alpha}(\eta_1) \cap C_{C\alpha}(\eta_2) \cap B(\xi_1, C2^{-k}) \cap B(\xi_2, C2^{-k}) \cap \mathscr{N}'_{k+k_1} \neq \emptyset$ that can be connected to each η_j by a path p with $\ell(p) \lesssim 2^{-k}$.

By lemma 8, the condition implies we may connect the η_j by a path of length $\lesssim 2^{-k}$. If we choose ε small enough, then if $B \subseteq NB'$, where $2^{-k} \leq |B'| \leq 2^{-k+1}$, $2 \leq N \leq 2M$, and $\beta(NB') < \varepsilon$ we may pick $\eta_j \in B(\xi_j, M\varepsilon 2^{-k})$ that satisfy (**C1**) as in the case of Flat-Bad balls (since our set is so straight locally, the cones $C_\alpha(\eta_j)$ will have large intersection for our choices of ξ_j) and connect those by a polygonal path. This exception will be needed in Case 2 of Lemma 19.

If no such B' exists for B, then just a pair η_1 and η_2 assured by (**C1**) . Note that such pairs might not exist in general, but we will show below that they exist often enough.

Remark 11. This is the only point in the proof that breaks down in infinite dimensions, since we are controlling the number of bridges we build for a ball $B \in \mathscr{B}_k$ by $\#(\Delta_k \cap MB)$, which is uniformly bounded so long as we work in finite dimensions. We do however conjecture that Theorem 1 still holds with constants independent of dimension and in fact holds in the case of an infinite dimensional Hilbert space.

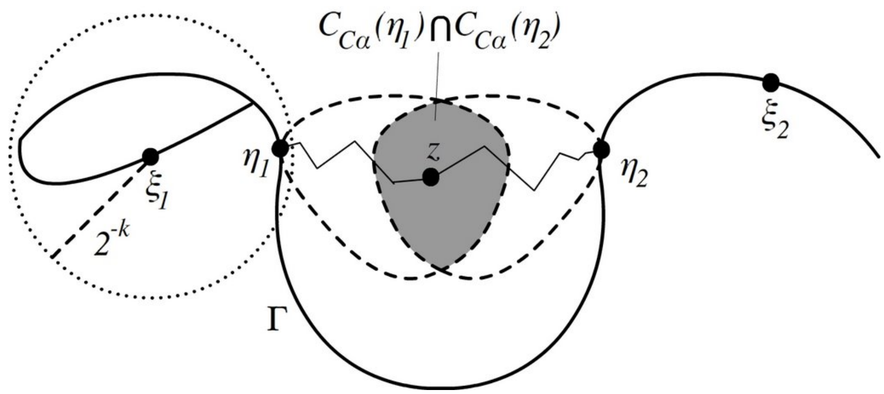

Figure 4: Building bridges from a Non-Flat ball.

3.3 Estimating the total length

Let $\tilde{\Gamma}$ be the union of Γ with all paths we have added for Flat-Bad cubes and Non-Flat balls. The goal of this section is to bound the length of $\tilde{\Gamma}$.

Lemma 12. *Let $\tilde{\Gamma}$ be as above. Then $\mathscr{H}^1(\tilde{\Gamma}) \lesssim_d \mathscr{H}^1(\Gamma)$.*

To prove this, we will need some additional lemmas. The following lemma and it's techniques will be used to help estimate the length we have added on from Flat-Bad cubes. It will also be used later, when we find short paths between points in $\tilde{\Gamma}$.

Lemma 13. *There exists an $\varepsilon_0 > 0$ and $C > 0$ such that for any $K > 0$ and any connected compact Γ contained in any Hilbert space satisfying*

(i) for any $r < |\Gamma|$ and $x \in \Gamma$, we have

$$\beta_\Gamma(B(x,r)) < \varepsilon_0,$$

(ii) for any $r < |\Gamma|$ and $x \in \Gamma$, we have that the Hausdorff distance between $B(x,r) \cap \Gamma$ and $B(x,r) \cap L$ is less than $4\varepsilon_0|B|$ for some line L.

(iii) for $x \in \Gamma$

$$\sum_{k=-\log|\Gamma|}^{\infty} \beta_\Gamma^2(x, 2^{-k}) < K\,,$$

we have

$$\mathscr{H}^1(\Gamma) \lesssim |\Gamma| e^{CK}.$$

Note that assumption (ii) is assured by Remark 7.

Remark 14. The assumptions (i) and (ii) above may be omitted, but that would require a longer, more complicated proof. This lemma does not appear in the literature for infinite dimensional Hilbert spaces. However, for the plane, this is shown in [3]. Also see Theorem X.2.1 in [8].

Proof. Without loss of generality, we assume $|\Gamma| = 1$. Fix C' such that $\frac{1}{10\varepsilon} > C' \geq 1$. We inductively construct a sequence of polygonal curves P_n as follows. Let $x_0, x_1 \in \Gamma$ be such that $\text{dist}(x_0, x_1) = |\Gamma|$. We set $P_0 = I_\emptyset = [x_0, x_1]$ and get $\ell(P_0) = 1$. By property (ii), we have the existence of $x_2 \in \mathscr{N}_{\varepsilon_0}[x_0, x_1]$, the ε_0 neighborhood of the interval $[x_0, x_1]$ satisfying $|x_1 - x_2| \sim_{C'} |x_0 - x_2|$. We set $P_1 = I_0 \cup I_1 = [x_0, x_2] \cup [x_2, x_1]$. By the Pythagorean theorem we have that

$$\ell(P_1) = |x_2 - x_0| + |x_1 - x_2| \leq 1 + C\beta^2_{\{x_0,x_2,x_1\}}(x_2, |x_0 - x_1|)\,.$$

The constant C here depends only on the choice of C' above. We will abuse notation and, for a triple x_0, x_1, x_2 as above, denote

$$\beta[x_0, x_1] := \beta^2_{\{x_0,x_2,x_1\}}(x_2, |x_0 - x_1|)\,.$$

We may now iterate this process on each of the intervals $[x_0, x_2]$ and $[x_2, x_1]$, getting a polygonal curve P_2 satisfying

$$\begin{aligned} \ell(P_2) &\leq \ell(I_0)(1 + C\beta^2[x_0, x_2]) + \ell(I_1)(1 + C\beta^2[x_2, x_1]) \\ &= \ell(P_1)\left(\frac{\ell(I_0)}{\ell(P_1)}(1 + C\beta^2(I_0)) + \frac{\ell(I_1)}{\ell(P_1)}(1 + C\beta^2(I_1))\right) \\ &\leq \ell(P_0)(1 + C\beta^2(I_\emptyset))\left(\sum_{i=0}^{1} \frac{\ell(I_i)}{\ell(P_1)}(1 + C\beta^2(I_i))\right). \end{aligned}$$

Continuing inductively, we get by at the n^{th} step a polygon with 2^n edges, satisfying

$$\ell(P_n) \leq \sum_{\omega \in \{0,1\}^n} a_\omega \Pi_{k=0}^n (1 + C\beta[I_{\omega_k}])$$

where ω_k is the truncation to the first k elements of ω, and $1 = \sum_{\omega \in \{0,1\}^n} a_\omega$. Note that

$$\Pi_{k=0}^n (1 + C\beta[I_{\omega_k}]) \leq e^{CK}\,.$$

Also note that, by property (i), at the n^{th} step, the vertices of P_n form a $\left(\frac{1+C'}{C'}\right)^{-n}$ net for Γ.

□

Remark 15. A few things things should be mentioned about the proof:

1. The choice of x_2 between x_0 and x_1 is not important so long as we pick it far from x_0 and x_1, i.e. $d(x_2, \{x_0, x_1\}) \gtrsim |x_0 - x_1|$. This ensures that our sequence of paths will converge to Γ.

2. In the construction in the proof, we could have stopped iterating at a finite polygonal path, or more generally, cease adjusting our sequence of curves on some collection of segments. The resulting path, by virtue of being a polygon or having corners, would not satisfy the conditions of the theorem at the vertices, however it would still satisfy the conclusion of the lemma.

3. Also note that condition (iii) can be replaced by

$$\sum_{kJ > -\log|\Gamma|} \beta_\Gamma^2(x, M2^{-kJ}) < K$$

since, for M large enough and ε_0 small enough, this will imply (iii) (with perhaps a different K).

We mention these facts since we will want to use the construction in this proof to construct polygonal paths with vertices in $\bigcup \Delta_{kJ}$ using the Flat-Bad condition on cubes.

Lemma 16. *Let $E(x,y)$ be the collection of maximal cubes Q with $|Q| \leq |S(x,y) \cap \Gamma|$, centers in $\overline{S(x,y)}$, and are not Flat-Good (so no cube in $E(x,y)$ is properly contained in another cube in $E(x,y)$). There exists $\varepsilon_0 > 0$ such that for any $x, y \in \Gamma$, if $\beta(R_{x,y}) < \varepsilon_0$, then*

$$\sum_{Q \in E(x,y)} |Q| \lesssim |x - y|.$$

Proof. Let $\Gamma_{x,y} = \overline{S(x,y)} \cap \Gamma$. Let $\mathscr{Q}_{x,y} = \{Q \in \mathscr{Q} : Q$ has center in $\Gamma_{x,y}$ and $|Q| \leq |\Gamma_{x,y}|\}$. Let k be the largest number for which there is no $Q \in \mathscr{Q}_k$ contained in $\Gamma_{x,y}$. Construct a path as in Lemma 13 as follows. For $j \geq 1$, choose points x_j and y_j in $\Delta_{(k+j)J} \cap \Gamma_{x,y}$ closest to x and y respectively (so $x_j \to x$ and $y_j \to y$ as $j \to \infty$).

Note that $[x_1, y_1] \cup \bigcup_{j \geq 1} [x_j, x_{j+1}] \cup [y_j, y_{j+1}]$ is a connected path connecting x to y. Let $P_0 = [x_1, y_1]$

Pick $x' \in \Delta_{(k+1)J} \cap \Gamma_{x,y}$ closest to the midpoint of $[x_1, y_1]$, replace P_0 with $P_1 = [x_1, x'] \cup [x', y_1]$. Continue this process, suppose we have a path P_n with

edge $[a,b]$, $a,b \in \Delta_{(k+1)J}$ and there is a point $x' \in \Delta_{(k+1)J}$ between a and b closest to the midpoint between a and b (so $d(x',\{a,b\}) \gtrsim |a-b|$). Then replace that edge with $[a,x'] \cup [x',b]$ to form P_{n+1}. If there is no such point between a and b, leave that edge.

In the end, we have constructed a path P^1 with vertices all points in $\Delta_{(k+1)J} \cap \Gamma_{x,y}$. Repeat the process as follows. Take an edge $[a,b]$, $a,b \in \Delta_{(k+1)J}$ such that either a or b is the center of a cube $Q \in \mathscr{Q}_{k+1} \cap \mathscr{Q}_{x,y}$ that is Flat-Good and is not the child of a Flat-Bad or Non-Flat cube in $\mathscr{Q}_{x,y}$. On this edge, perform the same as above, i.e. replace it with $[a,x'] \cup [x',b]$ where $x' \in \Delta_{(k+2)J} \cap \Gamma_{x,y}$ etc. This gives P^2. Continue inductively to get a sequence of paths P^n that converge to a path $P_{[x_1,y_1]}$. By Lemma 13 and Remark 15, $\ell(P_{[x_1,y_1]}) \lesssim |x_1 - y_1|$ if ε_0 is small enough (so that (i) of Lemma 13 is satisfied).

We can do similarly for each segment $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$ to make paths $P_{[x_j,x_{j+1}]}$ and $P_{[y_j,y_{j+1}]}$ respectively. Let

$$P = [x_1, y_1] \cup \bigcup_{j \geq 1} P_{[x_j,x_{j+1}]} \cup P_{[y_j,y_{j+1}]}$$

and

$$\ell(P) \lesssim |x_1 - y_1| + \sum_{j \geq 1} (|x_j - x_{j+1}| + |y_j - y_{j+1}|) \lesssim |x-y| \,,$$

where the last inequality is just the summing of a geometric sum. Note that the center a of any $Q \in E(x,y)$ is the endpoint of an edge $[a,b] \subseteq P$ of length $\ell([a,b]) \sim |Q|$. Let $e_Q = (a, \frac{a+b}{2})$. Then $\{e_Q : Q \in E(x,y)\}$ is a disjoint collection of segments in P with $\ell(e_Q) \sim |Q|$, and thus

$$\sum_{Q \in E(x,y)} |Q| \lesssim \ell(P) \lesssim |x-y|,$$

which proves the claim.

□

Proof of Lemma 12. Recall that any path p added on for a Non-Flat ball B has length $\lesssim |B|$. Thus, by Theorem 6, the total lengths of all paths p added may be bounded as follows.

$$\sum_{B \text{ Non-Flat}} \sum_{p \text{ path for } B} \ell(p) \lesssim \sum_{B \text{ Non-Flat}} |B| \leq \sum_{B \text{ Non-Flat}} \frac{\beta(MB)^2}{(\delta\varepsilon)^2} |B|$$
$$\lesssim \mathscr{H}^1(\Gamma).$$

For a Flat-Bad cube Q, let p_Q be the path constructed for the ball Q as described earlier. These have length $\lesssim |Q|$. Note that for each Flat-Bad cube Q, there is a chain $A(Q) = \{Q_{m+1}, ..., Q_k\}$ with $Q_k = Q$, $B_{m+1}, ..., B_{k-1}$ all Flat-Good , and satisfying (6). Let $D(Q)$ be the set of Flat-Bad descendants of Q that are connected to Q by a chain of Flat-Good cubes. Then

$$\sum_{Q \text{ Flat-Bad}} \ell(p_Q) \lesssim \sum_{Q \text{ Flat-Bad}} |Q| \leq \frac{1}{\varepsilon} \sum_{Q \text{ Flat-Bad}} \sum_{Q' \in A(Q)} |Q| \beta(MQ')^2$$
$$= \frac{1}{\varepsilon} \sum_{Q' \text{ Flat-Good}} \beta(MQ')^2 \sum_{Q \in D(Q')} |Q|.$$

We claim that

$$\sum_{Q \in D(Q')} |Q| \lesssim |Q'|,$$

in which case the lemma will follow by Theorem 6. This follows by applying Lemma 16 with x and y being points in Q' that maximize $|x - y|$. $\square$

4 Route finding: there are enough shortcuts

We now turn to proving that $\tilde{\Gamma}$ is quasiconvex, that is, any two arbitrary points $x, y \in \tilde{\Gamma}$ may be connected by a curve in $\tilde{\Gamma}$ of length $\lesssim |x - y|$. This is the final step in proving Theorem 1 (as well as Theorem 3).

In the first section, we reduce to the case when $x, y \in \Gamma$, using the properties in Lemma 8. After that, we state and prove the main lemma. Our main lemma says that between any points $x, y \in \Gamma$ there are bridges connecting points that are almost collinear with x and y. In the last part of this section, we pull these results together and conclude the main theorem.

4.1 The Case of x or y not in Γ

Lemma 17. *There is a universal constant $a > 0$ such that if s_1 and s_2 are two nonadjacent segments in $\tilde{\Gamma} \backslash \Gamma$, then $d(s_1, s_2) \geq a \cdot \min\{|s_1|, |s_2|\}$.*

Proof. Let $[x, y]$ and $[z, w]$ be two non-adjacent segments such that $|x - y| \geq |z - w|$, $x, y, z, w \in \mathscr{N}$, and suppose there are points x' and z' in each of these segments respectively such that $|x' - z'| = d([x, y], [z, w]) < a|z - w|$, where a is a small constant we will pick shortly. Set $r = |x - y|$.

Recall that by the definition of $\mathscr{N}$, $w, z \notin B(x, cr)$. If $x' \in B(x, \frac{c}{2} r)$, we know

$$x \in R(s, t) \subseteq R(z, w) \subseteq R_\lambda(z, w)$$

where $(s,t) = [w,z] \cap B(x,cr)$, but this contradicts Lemma 8. Hence, applying a similar argument for y, we may assume

$$d(x', \{x,y\}) > \frac{c}{2}|x-y| \tag{7}$$

The idea for the remainder of the proof is to shift the line $[w,z]$ so that it intersects $[x,y]$, in which case both lines will lie in a plane and we can prove the result more easily in this case. The general case will follow because the amount we needed to shift is very small since we are assuming that the lines come very close to each other. See Figure 5

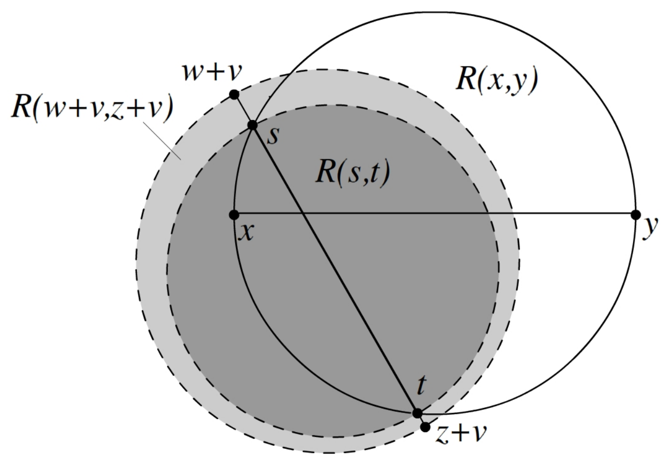

Figure 5:

Let $v = x' - z'$, so $|v| < a|z-w|$. By Lemma 8, $w, z \notin R_\lambda(x,y)$, and if a is much smaller than λ, $w+v, z+v \notin R_{\frac{\lambda}{2}}(x,y)$ as well. Now let $(s,t) = [w+v, z+v] \cap R(x,y)$, which, for a small enough with respect to c, is nonempty by (7). If $\frac{x+y}{2} \in (s,t)$, then

$$|w-z| = |(w+v)-(z+v)| \geq 2 \cdot \frac{1+\frac{\lambda}{2}}{2}|x-y| > |x-y|,$$

a contradiction. Hence (s,t) avoids the midpoint of $[x,y]$. Suppose now that x is closer to (s,t) than y. Then for a much smaller than λ,

$$x \in R(s,t) \subseteq R_{\frac{\lambda}{2}}(w+v, z+v) \subseteq R_\lambda(w,z),$$

again, a contradiction. $\square$

Corollary 18. *To prove the main theorem, it suffices to show that if $x, y \in \Gamma$ then they are joined by a path of length $\lesssim |x-y|$.*

Proof. If x, y are in the same segment or two adjacent segments in $\tilde{\Gamma}$, then we're done. If they are in two different segments s_x and s_y, then $|x-y| \geq a \min\{|s_x|, |s_y|\}$, but by (4), the endpoints of these segments are all of comparable distances between each other, so in fact $|x-y| \gtrsim \max\{|s_x|, |s_y|\}$. Let p_x and p_y be shortest paths connecting x and y to points $x', y' \in \Gamma$ respectively. These paths will have lengths $\sim |s_x|$ and $\sim |s_y|$ respectively by the construction (and Lemma 8), so $|x'-y'| \lesssim |x-y|$. Moreover, between x' and y', by assumption, there is a path p connecting them of length $\lesssim |x-y|$, and thus the path $p_x \cup p \cup p_y$ connects x and y and has length

$$\ell(p_x \cup p \cup p_y) \lesssim |s_x| + |x'-y'| + |s_y| \lesssim |x-y| + |x-y| + |x-y| \lesssim |x-y|.$$

Suppose $x \in \Gamma$ and $y \in \tilde{\Gamma} \backslash \Gamma$. There is a path of shortest length p_y connecting y to a point $y' \in \Gamma$ which has length $\ell(p_y) \sim d(y, \Gamma)) \leq \min\{|y-y'|, |y-x|\}$, and then a path $p_{xy'}$ connecting y' to x of length $\ell(p_{xy'}) \lesssim |x-y'|$. If $|x-y'| < 2|x-y|$, then

$$\ell(p_y \cup p_{xy'}) \lesssim |x-y| + |x-y'| < 3|x-y|$$

and otherwise, if $|x-y'| \geq 2|x-y|$, then

$$|x-y| \geq d(y, \Gamma) \sim |y-y'| \geq |x-y'| - |x-y| \geq |x-y|,$$

thus

$$|x-y| \sim d(y, \Gamma) \sim \ell(p_y) \sim |y-y'|,$$

hence

$$\ell(p_y \cup p_{xy'}) \lesssim |x-y| + |x-y'| \leq 2|x-y| + |y-y'| \lesssim |x-y|$$

and that finishes the proof.

□

4.2 Main Lemma

Lemma 19 (Main lemma)**.** *Let $x, y \in \Gamma$. Then we may find a path γ connecting x and y that is either a chord-arc path in Γ or a union of segments S, with endpoints in Γ and a set $P \subseteq \tilde{\Gamma}$ that satisfy*

$$\sigma = \sum_{s \in S} \ell(s) \leq |x-y| \tag{8}$$

and

$$\mathscr{H}^1(P) \lesssim |x-y| - \sigma. \tag{9}$$

In the proof that follows, we set several constants. Let us mention our order of choosing of the constants we use below so there is no ambiguity. Let φ be a small angle that we will fix later. Then we will set α in terms of φ, and C in terms of φ and α. We pick M to be large in terms of φ, M' in terms of M, δ in terms of M and M', and ε small depending on M and M'.

4.3 Proof of main lemma

Proof. We describe a process of constructing γ, i.e. obtaining P and S as in the statement of the lemma. We do so inductively. In particular, we will have a sequence of paths γ_j such that each path is a union of paths in $\tilde{\Gamma}$ and line segments, and the consecutive γ_{j+1} is constructed by replacing each of the segments with another path according to some schema.

Let L the infinite line through x and y, Π the projection onto this line, and $\varphi > 0$ some small angle to be chosen later, and $M' < M$ a large number to be chosen later (in fact, it will be picked proportionally to M). The schema for replacing a segment $[x, y]$ with a new path is organized into four cases:

4.3.1 $(x, y) \cap \Gamma \neq \emptyset$

Decompose $[x, y] \backslash \Gamma$ into subintervals $\{(a_i, b_i)\}$ and let γ be the path with segments $[a_i, b_i]$.

From now on, we assume $(x, y) \cap \Gamma = \emptyset$. Let $B \in \mathscr{B}$ be the smallest ball containing x and y. If $2^{-n} \leq |x - y| < 2^{-n+1}$, then $B \in \mathscr{B}_n \cup \mathscr{B}_{n+1}$.

4.3.2 $\beta(M'B) < \varepsilon$

Suppose first that $x, y \in \Delta_{kJ}$ are adjacent. We will construct a sequence of paths γ_n by adjusting or adding edges. When adjusting a γ_n to get γ_{n+1}, we may add some edges that will be permanent in the sense that they will be contained in γ_i for all $i > n$. We will keep track of these edges by placing them in a collection S.

Let $x = x_0, x_1, ..., x_n = y$ be the points in $\Delta_{(k+1)J}$ between x and y. Let γ_1' be the path obtained by connecting these points in order. If $[x_j, x_{j+1}]$ is an edge such that $Q = Q(x_j) \in \mathscr{Q}_{k+1}$ is not Flat-Bad, then since ε is small enough, there is a path $p_Q \subseteq \tilde{\Gamma}$ and segments e_Q^1 and e_Q^2 with $\ell(e_Q^i) \lesssim M\varepsilon|Q|$ such that $p_Q \cup e_Q^1 \cup e_Q^2$ is a path connecting x_j to x_{j+1}. Do similarly if $Q(x_{j+1})$ is not Flat-Bad. Add these edges to our set S. Doing this on each edge in γ_1' makes a new path γ_1.

Repeat the above process on each edge $[x_j, x_{j+1}]$ in γ_1' that remained in γ_1 (i.e. both $Q(x_j)$ and $Q(x_{j+1})$ are Flat-Good) to get a path γ_2 and so on to get a sequence of paths γ_n that converge to a path γ with length $\lesssim |x_0 - x_1| = |x - y|$ by Lemma 13 and the Remark. Moreover, $\gamma\backslash\tilde{\Gamma} = \bigcup_{s\in S} s^\circ$, where s° denotes the relative interior of s.

Recall that each $s \in S$ is associated to some maximal Flat-Bad or Non-Flat cube Q with $\ell(s) \lesssim M\varepsilon|Q|$ (that is, Q is contained in no other Flat-Bad or Non-Flat cube with an edge associated to it), and each such cube has no more than three edges associated to it. By Lemma 16, for ε small enough,

$$\sigma = \sum_{s\in S} \ell(s) \lesssim M\varepsilon \sum_{Q\in E(x,y)} |Q| \lesssim M\varepsilon|x - y|.$$

Let $P = \gamma \cap \Gamma$. Then we can pick ε small enough so that $\sigma < \varepsilon_1|x-y|$, where we pick $\varepsilon_1 < \frac{1}{4}$ below, and hence

$$\mathscr{H}^1(P) \leq \ell(\gamma) \lesssim |x - y| \lesssim |x - y| - \sigma.$$

Now suppose that x, y are arbitrary. Again, we will construct a collection S of edges. Choose k large enough so that if we order the points $x_0, ..., x_n \in \Delta_{kJ}$ between x and y by their distance from x, then $|x_0 - x| + |x_n - y| < \frac{1}{4}|x - y|$ and add $[x_0, x]$ and $[x_n, y]$ to S. Let $\gamma' = [x, x_0] \cup \bigcup_{j=0}^{n-1}[x_j, x_{j+1}] \cup [x_n, y]$. Then $\ell(\gamma') \lesssim |x - y|$. If any edge $[x_j, x_{j+1}]$ has $Q(x_j)$ or $Q(x_{j+1})$ not Flat-Good, as before there is a path $\gamma_j = p_Q \cup e_Q^1 \cup e_Q^2$ connecting x_j to y_j, add e_Q^1 and e_Q^2 to S. Let $S_j = \{e_Q^1, e_Q^2\}$ and $P_j = p_Q$. For ε small, these satisfy

$$\mathscr{H}^1(P_j) \lesssim |x_j - x_{j+1}| - \sigma_j, \quad \sigma_j = \sum_{s\in S_j} \ell(s) < \varepsilon_1|x_j - x_{j+1}|. \tag{10}$$

Otherwise, since $x_j, x_{j+1} \in \Delta_{kJ}$ are adjacent, we may apply the previous construction to these points (if ε is sufficiently small) to get a path γ_j that is the union of a set P_j and collection of segments S_j satisfying (10) as well. Add the segments from each such S_j segments to S.

Note there is a universal constant C_0 such that

$$|x - x_0| + |x_n - y| + \sum_j |x_j - x_{j+1}| \leq C_0|x - y|.$$

Let $\varepsilon_1 < \frac{1}{4C_0}$, then by (10),

$$\begin{aligned}\sigma = \sum_{s\in S}\ell(s) = |x-x_0| + |x_n - y| + \sum_j \sum_{s\in S_j}\ell(s) \\ < \frac{1}{4}|x-y| + \varepsilon_1 \sum |x_j - x_{j+1}| \\ < (\frac{1}{4}+\frac{1}{4})|x-y| = \frac{1}{2}|x-y|.\end{aligned}$$

Let

$$\gamma = [x, x_0] \cup [x_n, y] \cup \bigcup_j \gamma_j$$

and $P = \bigcup P_j$. Then

$$\mathscr{H}^1(P) \lesssim \sum_j |x_j - x_{j+1}| \lesssim |x-y| \lesssim |x-y| - \sigma$$

and γ thus satisfies the conditions of our lemma.

From now on, we assume $\beta(M'B) > \varepsilon$.

For $z \in [x, y]$, $2^{-k} < r \leq 2^{-k+1}$. define $r_z = \sup\{r : B(z,r) \cap \Gamma = \emptyset\}$, and fix $z \in (x, y)$ such that r_z is maximum.

Identify L with $\mathbb{R}$ so that $x > y$ and let $\Pi : \mathbb{R}^d \to \mathbb{R}$ be the orthogonal projection onto L. Define

$$H_z^- := \{v : y < \Pi(v) < z\}, \quad H_z^+ := \{v : z < \Pi(v) < x\}.$$

That is, $\mathscr{H}_z^+$ is the area trapped in between two parallel half planes, once centered at x and another at z, that are perpendicular to L. Similarly for $\mathscr{H}_z^-$. For $j = 1, 2$, define $V_j(z) = C_{\frac{1}{\sin j\varphi}, L}(z)^c$, that is, a cone centered at z with axis L.

Lemma 20. *For sufficiently small $\varphi > 0$, if*

$$x' \in H_z^+ \cap V_1(z)$$

and

$$y' \in H_z^- \cap V_1(z)$$

are connected by a chord-arc path p, then $\gamma = [x, x'] \cup p \cup [y, y']$ satisfies (8) *and* (9).

Proof. Clearly (8) is satisfied. Choose $\varphi > 0$ small enough so that $\sin\varphi < \frac{\cos\varphi}{2}$. Let $\varphi' < \varphi$ be the angle that $[x', z]$ makes with L and θ the angle $[x, x']$ makes with L. Then by the law of sines,

$$\begin{aligned}|x - x'|(1 - \cos\theta) &= \frac{\sin^2\theta}{1 + \cos\theta}|x - x'| = \frac{\sin\theta\sin\varphi'}{1 + \cos\theta}|x' - z| \le \sin\varphi'|x' - z| \\ &< \frac{\cos\varphi'}{2}|x' - z|\end{aligned}$$

and since $|x - x'|\cos\theta + |x' - z|\cos\varphi' = |x - z|$,

$$\begin{aligned}|x - z| - |x - x'| &= (\cos\theta - 1)|x - x'| + \cos\varphi'|x' - z| \ge \frac{\cos\varphi'}{2}|x' - z| \\ &= \frac{1}{2}|x'' - z|\end{aligned}$$

where $x'' = \Pi(x')$. Similarly,

$$|y - z| - |y - y'| \ge \frac{1}{2}|y - y''|$$

where $y'' = \Pi(y')$. Hence,

$$\begin{aligned}|x - y| - |x - x'| - |y - y'| &= |x - z| + |z - y| - |x - x'| - |y - y'| \\ &\ge \frac{1}{2}(|x - x''| + |y - y''|) = \frac{1}{2}|x'' - y''|.\end{aligned}$$

Note that the angle that $[x', y]$ makes with L can be no more than φ, thus

$$\ell(p) \lesssim |x' - y'| \le \frac{1}{\cos\varphi}|x'' - y''| \lesssim |x - y| - |x - x'| - |y - y'|$$

which proves (9).

□

Remark 21. By this Lemma, if we pick φ small so that $\sin 2\varphi < \frac{\cos 2\varphi}{2}$, it now suffices for us to find $x', y' \in \Gamma$ in each component of $V_2(z)$ that are connected by a chord arc path in $\tilde{\Gamma}$, which is what we'll do in the next two cases (see Figure 6).

4.3.3 $\beta(B(z, M'r_z)) < \varepsilon$

We need a proposition that will be used rather frequently in the arguments below:

Proposition 22. *Identify L with $\mathbb{R}$ and suppose a and b are points so that $\Pi(a) > \Pi(b)$ and the following hold:*

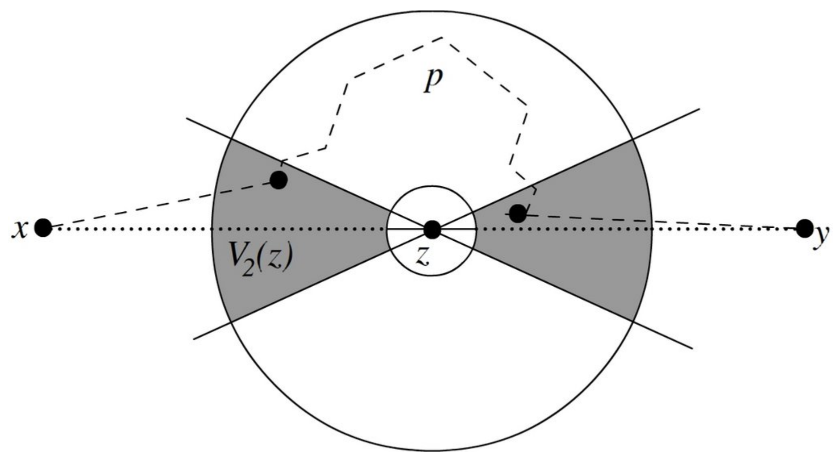

Figure 6: γ as constructed in cases 4.2.3 and 4.2.4.

(a) $\Pi(a) \in (x,y)$

(b) $a,b \in B(z_0, \frac{M'r_0}{4})$ *for some* $z_0 \in L$,

(c) $|a-b| > cr_0$, *c some small constant,*

(d) $\frac{dist(a,L)-dist(b,L)|}{|\Pi(a)-\Pi(b)|} \geq \frac{|dist(a,L)-dist(b,L)|}{|a-b|} > \frac{10}{M'}$.

(e) $\beta(B(z_0, M'r_0)) < \varepsilon$.

Let $z' = \Pi(a) + w + 2r_0$, *where*

$$w = \begin{cases} \frac{2r_0(\Pi(a)-\Pi(b))}{dist(a,L)-dist(b,L)} & \Pi(a) > \Pi(b) \\ 0 & \text{otherwise} \end{cases}.$$

Then $z' \in B(z_0, \frac{M'r_0}{2})$ *and* $dist(z',\Gamma) > \frac{3}{2}r_0$ *for small* ε *(depending on c). Furthermore,* $x, y \notin B(z_0, M'r_0) \cap [\Pi(a), \infty)$, *so in particular,* $z' \in [x,y]$. *(See Figure 7).*

Proof. First consider the case when L and $L_{a,b}$ lie in the same two dimensional plane. Identify this plane with $\mathbb{R}^2$ so that $L = \mathbb{R}$, $\Pi(a) \geq \Pi(b)$, and a lies in the upper half plane. Suppose $\Pi(a) > \Pi(b)$. Let $f(t) = \text{dist}(t, L_{a,b})$, which, in this case, is a linear function with slope at least $\frac{10}{M'}$. Hence

$$f(\Pi(a) + w) \geq f(w) = 2r_0$$

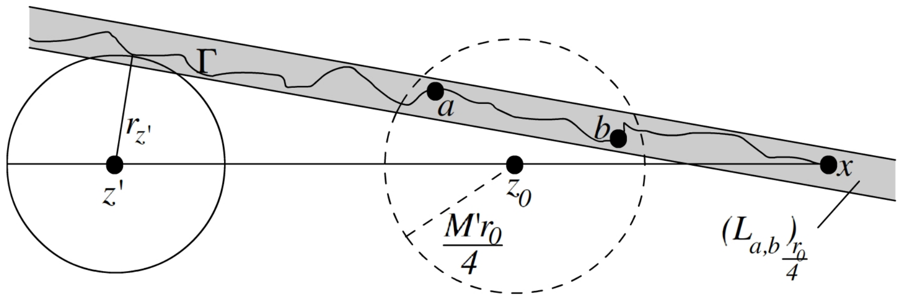

Figure 7: The heuristic for Proposition 22 is that if two points a and b satisfy these conditions, then Γ is contained in a tube around $L_{a,b}$ that is moving away from L, so somewhere along L there must be a point far away from $L_{a,b}$ and hence from Γ.

and so $\Pi(a) + w + 2r_0$ is at least $2r_0$ from $L_{a,b}$. Thus

$$\text{dist}(L_{a,b}, B(\Pi(a) + w + 2r_0, \frac{3}{2}r_0)) > \frac{1}{4}r_0.$$

Moreover, since $\Pi(a) \in B(z_0, \frac{M'r_0}{4})$,

$$|z' - z_0| \leq \frac{M'r_0}{4} + |w| + 2r_0 \leq r_0(\frac{M'}{4} + \frac{2M'}{10} + 2) < \frac{M'}{2}r_0$$

for large enough M', hence

$$\text{dist}(B(z', \frac{3}{2}r_0), L_{a,b} \cup \partial B(z_0, M'r_0)) \geq (\frac{M'}{2} - \frac{3}{2})r_0 > \frac{1}{4}r_0. \tag{11}$$

Therefore, for small enough $\varepsilon > 0$, by property (e), $\Gamma \cap B(z_0, M'r_0)$ is contained in a tube of radius no more than $2M'r_0\varepsilon$, and by property (c), for small enough $\varepsilon > 0$ this tube is contained in $(L_{a,b})_{\frac{r_0}{4}}$, and thus we have $\text{dist}(z', \Gamma) > \frac{3}{2}r_0$. The case $\Pi(a) = \Pi(b)$ can be treated similarly.

Now consider the case when L and $L_{a,b}$ lie in two parallel hyperplanes H and $H_{a,b}$ in $\mathbb{R}^d$. Let $a', b', L'_{a,b} = L_{a',b'}$ be the projections of a, b, and $L_{a,b}$ onto the hyperplane containing L . Since this projection is isometric, $\Pi(a) = \Pi(a'), \Pi(b) = \Pi(b')$, and $\text{dist}(a, L) - \text{dist}(b, L)$ decreases as $H_{a,b}$ moves away from H, properties (a) through (d) of the proposition still hold with a', b' instead of a, b. Thus the estimate 11holds with z' as before since it holds with a' and b' in place of a and b. The proof that $\text{dist}(z', \Gamma) \geq \frac{3}{2}r_0$ is similar to the previous case.

To prove the final statement, notice that if $x \in B(z_0, M'r_0) \cap [\Pi(a), \infty)$, then

$$f(x) \geq f(\Pi(a)) = \text{dist}(a, L) \geq \text{dist}(a, L) - \text{dist}(b, L) > |a - b|\frac{10}{M'} > \frac{10cr_0}{M'},$$

which means that x can't be contained in $(L_{a,b})_{\frac{r_0}{4}}$ if we pick ε small enough.

□

In particular, we have the following useful corollary:

Corollary 23. *If $r_0 \geq r_z$, there are no points $a, b \in \Gamma$ satisfying (a)-(e).*

Proof. This is simply because $r_{z'} > r_z$ contradicts the maximality of r_z. □

Corollary 24. *If $z_0 \in (x, y)$ and $\beta(B(z_0, M'r_0)) < \varepsilon$, $r_0 \geq r_z$, then $x, y \notin B(z_0, M'r_0 2^{-4})$.*

Proof. Suppose $x \in B(z_0, M'r_0 2^{-4})$ (the case with y is identical). Let ξ be a point in $B(z_0, M'r_{z_0})$ that projects onto z_0, so $|z_0 - \xi| < 2M'r_0\varepsilon$. Then for ε small enough,

$$|\xi - x| \geq \mathrm{dist}(\xi, L) \geq \mathrm{dist}(z_0, \Gamma) = r_0,$$

$$\frac{\mathrm{dist}(\xi, L) - \mathrm{dist}(x, L)}{|\xi - x|} \geq \frac{r_0 - 0}{|\xi - z_0| + |z_0 - x|} \geq \frac{r_0}{2M'r_0 + M'r_0 2^{-4}} > \frac{10}{M'}.$$

Hence ξ and x satisfy the conditions of Proposition 22, and there exists a point $z' \in [x, y] \cap B(z_0, \frac{M'r_{z_0}}{2}) \cap [x, y]$ that is at least $\frac{3}{2}r_{z_0} > r_z$ from Γ, contradicting the maximality of r_z. □

Let m be such that $B \in \mathscr{B}_m$. There is $\xi_k \in B(z, 2r_z) \cap \Delta_k$ where $2^{-k} \leq r_z < 2^{-k+1}$. By Corollary 24 applied to $z_0 = z$, $z_k = \Pi(\xi_k) \in (x, y)$. Since $\beta(M'B) > \varepsilon$ and $\xi_k \in B$, there is a sequence $\{\xi_j\}_{j=m}^k$ such that ξ_m is the center of B, $\xi_j \in \Delta_j$ and $\xi_j \in B(\xi_{j-1}, 2^{-j})$ and furthermore, there must be a smallest ball $B(\xi_j, 2^{-j})$, $m \leq j \leq k$ such that $\beta(B(\xi_j, M'2^{-j})) > \varepsilon$ and $x, y \notin B(\xi_j, 2^{-j})$. Let z_i denote the projection of ξ_i onto L for $j \leq i \leq k$, which are contained in (x, y) again by Corollary 24.

Corollary 25. $|\xi_i - z_i| < 2^{-i+2}$ *for $j \leq i \leq k$.*

Proof. Let $\xi_j' \in \Gamma$ be such that $|\xi_j' - z_j| = r_{z_j} < r_z$. We prove this by induction. If $j = k$, then

$$|\xi_k - z_k| \leq \sqrt{|\xi_k - z_k|^2 + |z_k - z|^2} = |\xi_k - z| < 2r_z.$$

Suppose the claim is true down to some $i > j$ but not true for $i - 1$. Then since $\xi_{i-1}, \xi_i \in \Delta_i$,

$$|\xi_{i-1} - \xi_i| \geq 2^{-i}.$$

Furthermore,

$$|\xi_{i-1} - z_i| \leq |\xi_{i-1} - \xi_i| + |\xi_i - z_i| \leq 2^{-i+1} + 2r_z < 2^{-i+1} + 2^{-k+2}$$
$$< 2^{-i+1} + 2^{-i+2} = 2^{-i+3}.$$

Hence $\xi_{i-1}, \xi_i \in B(z_i, \frac{M'2^{-i}}{4})$ if we pick $M' > 2^5$. Furthermore, by the choice of j, since $i > j$, we know $\beta(B(z_i, M'2^{-i}) < \varepsilon$, and since

$$|z_i - z_{i-1}| \leq |\xi_i - \xi_{i-1}| \leq 2^{-i},$$

we have

$$\frac{\text{dist}(\xi_{i-1}, L) - \text{dist}(\xi_i, L)}{|\xi_{i-1} - \xi_i|} \geq \frac{2^{-i+2} - r_z}{2^{-i+1}} \geq \frac{2^{-i+1}}{2^{-i+1}} = 1,$$

hence ξ_{i-1}, ξ_i satisfy the the conditions of Proposition 22, so there is a $z' \in B(z_i, \frac{M'2^{-i}}{2})$ with $r_{z'} > \frac{3}{2}2^{-i} > r_z$, a contradiction. The second inequality in the corollary follows from the way we picked k. □

Claim: :

$$(B(z_j, 2^{-j+4} \csc \varphi) \backslash B(z_j, 2^{-j+3} \csc \varphi)) \cap \Gamma \subseteq V_1(z_k).$$

We assume $j < k$, since the case of $j = k$ can be proven in a similar manner. Suppose there is a point $\xi \in \Gamma$ in the annulus but outside the cone. Then

$$\text{dist}(\xi, L) \geq 2^{-j+3} \csc \varphi \sin \varphi = 2^{-j+3}. \tag{12}$$

Then, for $M' > 40(2^4 \csc \varphi + 3)$, and since our choice of φ gives $\csc \varphi > 2$,

$$|\xi - \xi_{j+1}| \geq |\xi - z_{j+1}| - |z_{j+1} - \xi_{j+1}| \geq 2^{-j+3} \csc \varphi - 2^{-j+3} \geq 2^{-j+4} - 2^{-k+3} \geq 2^{-j+3},$$

$$|\xi - z_{j+1}| \leq 2^{-j+4} \csc \varphi < \frac{M2^{-j-1}}{4},$$

and

$$\begin{aligned}
\frac{\text{dist}(\xi, L) - \text{dist}(\xi_{j+1}, L)}{|\xi - \xi_{j+1}|} &\geq \frac{2^{-j+3} - 2r_z}{|\xi - z_j| + |z_j - z_{j+1}| + |z_{j+1} - \xi_{j+1}|} \\
&\geq \frac{2^{-j+3} - 2^{-k+2}}{2^{-j+4} \csc \varphi + 2^{-j} + r_z} \geq \frac{2^{-j-2}}{2^{-j}(2^4 \csc \varphi + 1 + 2)} \\
&\geq \frac{10}{M'}.
\end{aligned}$$

Hence $\xi, \xi_{j+1} \in B(z_{j+1}, \frac{M'2^{-j-1}}{4})$ and $\beta(B(z_{j+1}, M'2^{-j-1}) < \varepsilon$ by our choice of j, thus ξ and ξ_{j+1} satisfy conditions (a)-(e) of the proposition, which is a contradiction, giving us the claim.

Now, fix k' so that $2^{-k'} \leq \frac{1}{2}|1-e^{i\varphi}| < 2^{-k'+1}$. Then we may find points $\eta_i \in \Delta_{j+k'}$, $i = 1, 2$, on either component of the cone $V_2(z_j)$ such that $B(\eta_i, M2^{-j-k'}) \supseteq B(\xi_j, M'2^{-j})$, which is true if we fix $M > M'2^{k'}$

$$\beta(B(\eta_i, M2^{-j-k'})) > \frac{M'2^{-j}}{M2^{-j-k'}}\beta(B(\xi_j, M'2^{-j})) > \delta\varepsilon$$

for $i = 1, 2$, if we pick $\delta < \frac{M'}{M2^{-k'}}$. Hence the balls $B(\eta_i, 2^{-j-k'})$ are Non-Flat and satisfy (**C1**) (so long as we pick $C > \frac{2 \cdot 2^{-j+4}\csc\varphi}{2^{-(j+k')}}$) since both points are in $B(z_j, 2^{-j+2})$), hence there are $\eta_i' \in B(\eta_i, M\varepsilon 2^{-j-k'})$ that are contained in the two components of $V_2(z_j)$ that are connected by a chord-arc path $p \subseteq \tilde{\Gamma}$ with length $\lesssim 2^{-j}$ as desired. (See Figure 8.) This follows from our choice of k' and that

$$\begin{aligned}\operatorname{dist}(\eta_i', \Gamma \cap (B(z_j, 2^{-j+4}\csc\varphi) \backslash B(z_j, 2^{-j+3}\csc\varphi))) &\leq \operatorname{dist}(\eta_i', \eta_i) + 2^{-j-k'} \\ &< (M\varepsilon + 1)2^{-j-k'} < 2 \cdot 2^{-j}\frac{1}{2}|1-e^{i\varphi}| = 2^{-j}|1-e^{i\varphi}|\end{aligned}$$

hence $\eta_i - z_j$ makes an angle of no more than 2φ with L.

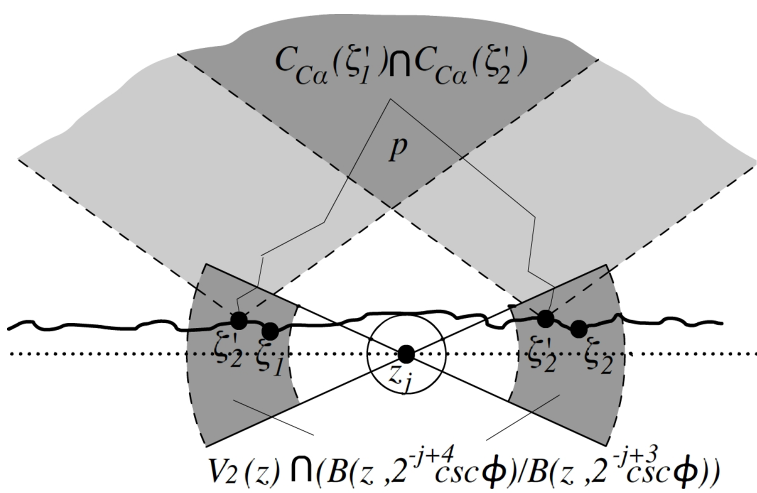

Figure 8: The η_j satisfy (**C1**) since $\beta(\xi_{j+1}, M'2^{-j-1}) < \varepsilon$.

4.3.4 $\beta(B(z, M'r_z)) \geq \varepsilon$

Claim: : There exist $\xi_\pm \in \Gamma \cap V_1(z) \cap H_z^\pm \cap B(z, \alpha r_z)$ with $z \in C_\alpha(\xi_j)$.

It suffices to find $\xi = \xi_+$, since the proof is the same for ξ_-. Suppose there was no such point. Let $\xi \in \Gamma \cap V_1(z) \cap \mathscr{H}_z^+ \cap B(z, \alpha r_z)$ be closest to z_j. Then since z cannot be in $C_\alpha(\xi_j)$,

$$|z - \xi| \geq \alpha \text{dist}(z, \Gamma) = \alpha r_z,$$

thus $\Gamma \cap V_1(z) \cap H_z^+ \cap B(z, \alpha r_z) = \emptyset$. For large enough α, however, we may find another ball centered on $\mathscr{H}_z^+ \cap [x, y]$ with radius larger than r_z, which is a contradiction (see Figure 9).

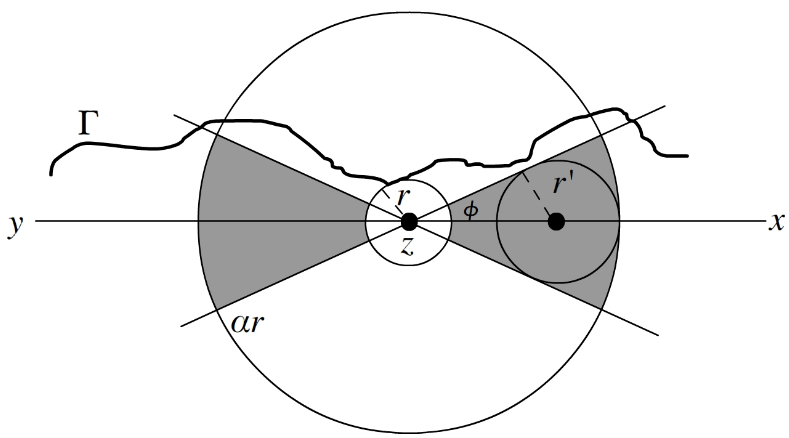

Figure 9: If $\Gamma \cap V_1(z) \cap H_z^+ \cap B(z, \alpha r_z) = \emptyset$, for large α we may find a ball in the cone with radius larger than r_z and disjoint from Γ.

Let ξ_+ be the point in $\Gamma \cap V_1(z) \cap \mathscr{H}^+ \cap B(z, \alpha r_z)$ closest to z that satisfies the claim. Identify the two-dimensional plane containing x, z, and ξ_+ with $\mathbb{C}$ so that $z = 0$, $x > 0$, and $\Im \xi_+ \geq 0$.

We may pick C' large enough so that $C_{C'\alpha}(\xi_+)$ contains the path $[\xi_+, t] \cup [t, z]$, where $t \in \mathbb{R}$ is such that $\arg(\xi_+ - t) \geq \frac{\pi}{4}$, and then C'' larger so that $B(z, \frac{r_z}{2}) \subseteq C_{C''\alpha}(\xi_+)$. Note that we may pick C'' independent of ξ_+ and z.

Pick $x' \in B(\xi_+, 2^{-k'}) \cap \Delta_{k'}$, where $k' = k'(r_z)$ is now the integer such that

$$2^{-k'} \leq \frac{1}{2}|r_z e^{i\varphi} - r_z e^{i2\varphi}| = \frac{1}{2} r_z |1 - e^{i\varphi}| < 2^{-k'+1}.$$

We may find y' and a ξ_- satisfying similar conditions in $\mathscr{H}_z^-$, and clearly $|x' - y'| < C'''2^{-k'}$ for some constant C'''.

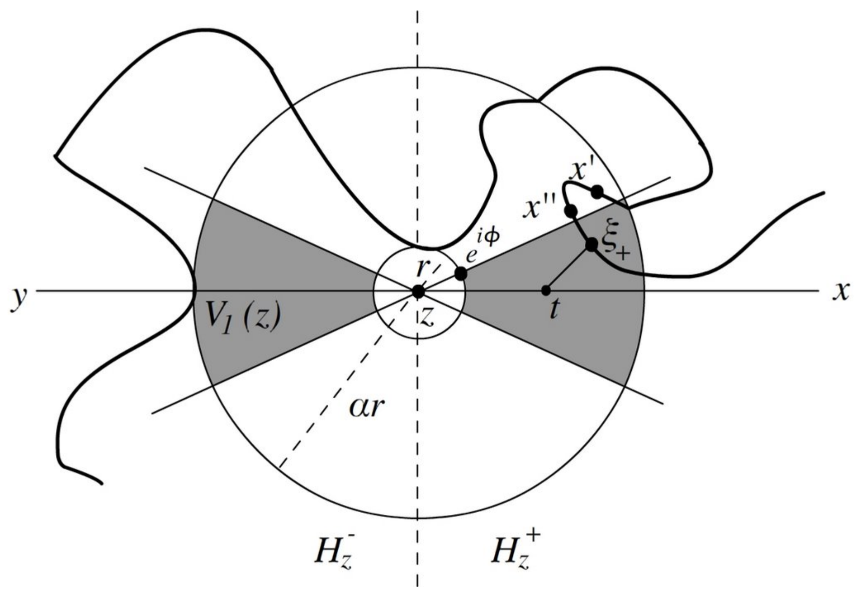

Figure 10:

Now fix k_1 so that there is $w \in B(z, \frac{r_z}{2}) \cap \mathscr{N}'_{k'+k_1}$ (recall the relationship between k' and r, so we can pick k_1 independent of these quantities). Then we may connect ξ_+ to w via the path $[\xi_+, t] \cup [t, z] \cup [z, w]$, which las length $\lesssim 2^{-k'}$, and a similar path can be found for ξ_-. Hence x' and y' satisfy the condition (**C1**) if we pick $C > \max\{C'', C'''\}$. Also,

$$\beta(B(x', M2^{-k'})) \geq \frac{M' r_z}{M2^{-k'}} \beta(B(z, M' r_z)) > \delta\varepsilon$$

so $B(x', 2^{-k'})$ is Non-Flat if we pick $\delta < \frac{4M'}{M|1-e^{i\varphi}|}$). Hence, we know that there are $x'' \in B(x', 2^{-k'})$ and $y'' \in B(y'', 2^{-k'})$ that are connected by a polygonal path of length $\lesssim 2^{-k'}$, and our choice of k' guarantees that $x'', y'' \in V_2(z) \cap H_z^+$, as desired.

□

4.4 Putting it all together

Using the above lemma, we can fully describe the construction of the curve connecting x and y. We run $[x, y]$ through our schema to get a new curve γ_1, which is

a union of segments S_1 and a set P_1 that satisfy

$$\sigma_1 = \sum_{s \in S_1} \ell(s) < |x - y|$$

and

$$\mathscr{H}^1(P_1) \lesssim |x - y| - \sigma_1.$$

On each segment $s \in S_1$, we replace each of them with new paths γ_s, which are unions of segments $s' \in S_s$ and a set P_s such that

$$\sigma_s = \sum_{s' \in S_s} \ell(s') \leq \ell(s)$$

and

$$\mathscr{H}^1(P_s) \leq C_1(\ell(s) - \sigma_s).$$

After replacing each of these segments, we form a new path γ_2 connecting x and y that is a union of segments S_2 and a set $P_2 = P_1 \cup \bigcup_{s \in S_1} P_s$ such that

$$\sigma_2 = \sum_{s \in S_2} \ell(s) = \sum_{s \in S_1} \sum_{s' \in S_s} \ell(s') \leq \sum_{s \in S_1} \ell(s) < |x - y|$$

and

$$\begin{aligned} \mathscr{H}^1(P_2) &\leq \mathscr{H}^1(P_1) + \sum_{s \in S_1} \mathscr{H}^1(P_s) \leq C_1(|x - y| - \sigma_1) + \sum_{s \in S_1} C_1(\ell(s) - \sigma_s) \\ &= C_1|x - y| - C_1 \sum_{s \in S_1} \sigma_s = C_1 \left(|x - y| - \sum_{s \in S_2} \ell(s) \right) \\ &= C_1(|x - y| - \sigma_2). \end{aligned}$$

Inductively, we may construct a sequence of curves γ_n such that each γ_n is a union of segments S_n and a set P_n such that

$$\sigma_n := \sum_{s \in S_n} \ell(s) < |x - y|$$

and

$$\mathscr{H}^1(P_n) \leq C_1(|x - y| - \sigma_n).$$

Hence, they converge to a Lipschitz curve γ that is contained in $\tilde{\Gamma}$ that satisfies $\ell(\gamma) \leq (C_1 + 1)|x - y|$. This combined with Lemma 18 and Lemma 12 finishes the proof of the main theorem.

References

[1] Jonas Azzam, *A chord-arc covering theorem in hilbert space*, Submitted.

[2] Christopher Bishop, *Tree-like decompositions of simply connected domains*, Preprint.

[3] Christopher J. Bishop and Peter W. Jones, *Harmonic measure, L^2 estimates and the Schwarzian derivative*, J. Anal. Math. **62** (1994), 77–113. MR MR1269200 (95f:30034)

[4] Michael Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), no. 2, 601–628. MR MR1096400 (92k:42020)

[5] Gautam Das and Giri Narasimhan, *Short cuts in higher dimensional space*, Proc. of the 7th Canadian Conference on Computational Geometry, 1995, pp. 103–108.

[6] Guy David, *Opérateurs d'intégrale singulière sur les surfaces régulières*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 2, 225–258. MR MR956767 (89m:42014)

[7] John B. Garnett, Peter W. Jones, and Donald E. Marshall, *A Lipschitz decomposition of minimal surfaces*, J. Differential Geom. **35** (1992), no. 3, 659–673. MR MR1163453 (93d:53010)

[8] John B. Garnett and Donald E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2008, Reprint of the 2005 original. MR MR2450237 (2009k:31001)

[9] Peter W. Jones, *Rectifiable sets and the traveling salesman problem*, Invent. Math. **102** (1990), no. 1, 1–15. MR MR1069238 (91i:26016)

[10] Claire Kenyon and Richard Kenyon, *How to take short cuts*, Discrete Comput. Geom. **8** (1992), no. 3, 251–264, ACM Symposium on Computational Geometry (North Conway, NH, 1991). MR MR1174357 (93h:65184)

[11] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability. MR MR1333890 (96h:28006)

[12] Joseph S. B. Mitchell, *Shortest paths and networks*, Handbook of Discrete and Computational Geometry (2nd Edition) (Jacob E. Goodman and Joseph O'Rourke, eds.), Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. 607–641.

[13] Giri Narasimhan and Michiel Smid, *Geometric spanner networks*, Cambridge University Press, Cambridge, 2007. MR MR2289615 (2009b:68002)

[14] Kate Okikiolu, *Characterization of subsets of rectifiable curves in* $\mathbf{R}^n$, J. London Math. Soc. (2) **46** (1992), no. 2, 336–348. MR MR1182488 (93m:28008)

[15] Raanan Schul, *Subsets of rectifiable curves in Hilbert space—the analyst's TSP*, J. Anal. Math. **103** (2007), 331–375. MR MR2373273 (2008m:49205)